Escape angles in bulk $\chi^{(2)}$ soliton interactions

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We develop a theory for non-planar interaction between two identical type I spatial solitons propagating at opposite, but *arbitrary transverse angles* in quadratic nonlinear (or so-called $\chi^{(2)}$) bulk media. We predict quantitatively the outwards escape angle, below which the solitons turn around and collide, and above which they continue to move away from each other. For in-plane interaction the theory allows prediction of the *outcome of a collision* through the inwards escape angle, i.e. whether the solitons fuse or cross. We find an analytical expression determining the inwards escape angle using Gaussian approximations for the solitons. The theory is verified numerically.

Stable self-guided laser beams or optical bright spatial solitons are of substantial interest in basic physics [1] and for technical applications, such as inducing fixed [2] and dynamically reconfigurable waveguides [3]. Several types of spatial solitons have been demonstrated experimentally, including 1D Kerr solitons [4] and 1D and 2D (number of transverse dimensions) solitons in saturable [5], photorefractive [6], and $\chi^{(2)}$ media [7]. Even incoherent solitons, excitable by a light bulb, have been demonstrated in photorefractive media [8]. All solitons exist when diffraction is balanced by the nonlinear self-focusing effect. In bulk Kerr media the self-focusing effect dominates and leads to collapse of both coherent and incoherent 2D solitons [9], their existence requiring an effectively saturable nonlinearity.

An intriguing feature of solitons is their particle like behavior during collision. In 1D Kerr media collisions are fully elastic due to integrability of the 1D nonlinear Schrödinger (NLS) equation [10]. In contrast, saturable, photorefractive, and $\chi^{(2)}$ media are described by non-integrable equations and soliton collisions are therefore inelastic, displaying both fusion (Fig. 1:A), crossing (Fig. 1:B), repulsion, and annihilation, additional solitons can be generated in a fission-type process [11], and solitons can even spiral around each other [12]. All processes depend strongly on the relative phase and have been demonstrated experimentally (see [1] and references therein).

Complex waveguide structures can be generated by soliton interaction, such as directional couplers [13], but their efficient implementation requires a detailed understanding of the nature of soliton collisions. Snyder and Sheppard predicted the outcome of collisions of 1D solitons in saturable media by comparing the collision angle with the critical angle for total internal reflection in an equivalent waveguide [14]. Except for this work most theories are based on the variational approach, which require the solitons to be far apart and breaks down at collision.

Here we focus on $\chi^{(2)}$ materials [15], which are more general than the simpler cubic Kerr and saturable media in the sense that dependent on the phase-mismatch between the fundamental and second-harmonic (SH) waves the nonlinearity can be both purely quadratic (close to phase-matching) and effectively cubic (for a large phase-mismatch). Spatial solitons in $\chi^{(2)}$ materials do not modify the refractive index, and consist of one (type I) or two (type II) fundamental fields resonantly coupled to a SH. The $\chi^{(2)}$ materials are of significant interest to photonics due to the strong and fast nonlinearities they can provide through cascading [16]. Furthermore, soliton induced waveguides in photorefractives can have a strong $\chi^{(2)}$ nonlinearity, which can be used for second-harmonic generation (SHG) [17].

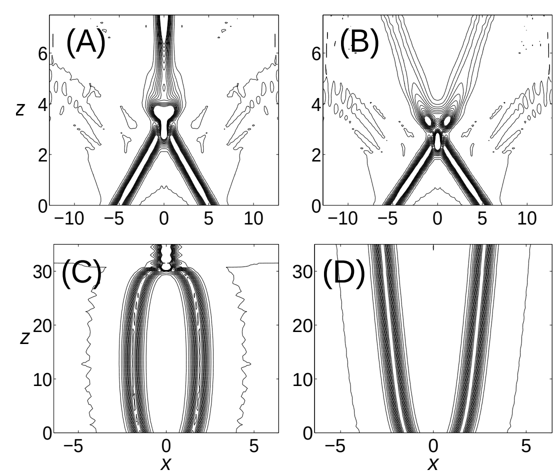

FIG. 1. A,B: Planar collision between two Gaussians, $P = 48.3$ and $\beta = 0$. C,D: Outwards launched exact solitons, $P = 122.4$ and $\beta = 5$. The launch angles are $\alpha_x = 58°$ (A), $\alpha_x = 62°$ (B), $\alpha_x = 5.4°$ (C), and $\alpha_x = 5.7°$ (D).

Fusion and crossing (Fig. 1:A,B) of spatial $\chi^{(2)}$ solitons has been demonstrated numerically [18] and experimentally [19], and fission of 1D type I solitons was demonstrated analytically and numerically in the large phase-mismatch limit approximately described by the NLS equation [20]. However, the $\chi^{(2)}$ system is more gen-

eral and complex than the saturable NLS equation and so far variational theories were only able to predict critical launch angles and relative phases separating regimes of collision and no collision [21]. Elegant non-planar effective particle theories predicted the absence of spiraling type I solitons, but still required weakly overlapping solitons [22]. In this letter we extend the effective particle approach to *arbitrary launch angles* and present the first theory able to correctly predict the *outcome of collisions* between 2D type I solitons in $\chi^{(2)}$ media.

We consider beam propagation under type-I SHG conditions in lossless bulk $\chi^{(2)}$ materials. Neglecting walk-off the system of normalized dynamical equations for the slowly varying envelope of the fundamental wave, $E_1 = E_1(\vec{r})$, and its SH, $E_2 = E_2(\vec{r})$, are [23]

$$i\partial_z E_1 + \tfrac{1}{2}\nabla_\perp^2 E_1 + E_1^* E_2 = 0, \tag{1a}$$

$$i\partial_z E_2 + \tfrac{1}{4}\nabla_\perp^2 E_2 - \beta E_2 + E_1^2 = 0. \tag{1b}$$

Here $\vec{r}$=(x, y, z), z is the propagation variable, and $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$ accounts for diffraction in the transverse $\vec{r}_\perp = (x, y)$-plane. The normalized phase mismatch is $\beta = l_d(2k_1 - k_2)$, where l_d is the diffraction length of the fundamental and k_1 and k_2 are the wave numbers of the fundamental and SH, respectively. The system (1) can be derived from the Lagrangian density

$$\mathcal{L} = 2\mathrm{Im}\left(E_1\partial_z E_1^*\right) + \mathrm{Im}\left(E_2\partial_z E_2^*\right) + \beta|E_2|^2 + |\nabla_\perp E_1|^2 + \tfrac{1}{4}|\nabla_\perp E_2|^2 - \mathrm{Re}\left(E_2^* E_1^2\right), \tag{2}$$

and conserves the power P=$\int(|E_1|^2$+$|E_2|^2)d\vec{r}_\perp$ and momentum $\vec{M}$=$\int \mathrm{Im}\{E_1^*\nabla_\perp E_1 + \frac{1}{2}E_2^*\nabla_\perp E_2\}d\vec{r}_\perp$, where we have defined $\int d\vec{r}_\perp \equiv \int\int_{-\infty}^{\infty} dxdy$.

The system (1) is known [24,25] to have a *one-parameter family* of radially symmetric bright 2D solitons of the form $E_1(\vec{r}) = V(r;\lambda)\exp(i\lambda z)$ and $E_2(\vec{r}) = W(r;\lambda)\exp(i2\lambda z)$ where $\lambda > \max(0; -\beta/2)$ is the internal soliton parameter and $r = \sqrt{x^2 + y^2}$. We have found this family numerically, using a standard relaxation method, and approximately, using the variational approach [25] with Gaussians profiles $(V, W) = (V_g, W_g)$,

$$V_g = a_1\exp(-r^2/b),\ W_g = a_2\exp(-r^2/b). \tag{3}$$

Here $a_1 = a_2[2(\lambda b - 1)]^{-\frac{1}{2}}$, $a_2 = \frac{3}{2}(\lambda + b^{-1})$, and $b = [1 + (12\lambda^2 + 8\lambda\beta + \beta^2)^{\frac{1}{2}}/(2\lambda + \beta)]/2\lambda$. Because the system is Galilean invariant we can apply a gauge transformation to find moving solitons. Thus the *general three parameter soliton family* $(\tilde{V}, \tilde{W})$ is given by

$$\tilde{V}(x - \nu_x z, y - \nu_y z; \lambda, \nu_x, \nu_y) = V(r; \lambda - \tfrac{1}{2}\nu_x^2 - \tfrac{1}{2}\nu_y^2)\exp[-i(\nu_x x + \nu_y y)], \tag{4a}$$

$$\tilde{W}(x - \nu_x z, y - \nu_y z; \lambda, \nu_x, \nu_y) = W(r; \lambda - \tfrac{1}{2}\nu_x^2 - \tfrac{1}{2}\nu_y^2)\exp[-2i(\nu_x x + \nu_y y)], \tag{4b}$$

where (V, W) are either the exact soliton profiles (V_s, W_s) found numerically or (V_g, W_g) given by (3). $\nu_{x,y} = \tan(\alpha_{x,y})$ are the initial transverse velocities corresponding to the launch angles $\alpha_{x,y}$ with respect to the z-axis.

We substitute a field composed of two weakly overlapping solitons $(\tilde{V}^{(i)}, \tilde{W}^{(i)})$ into the Lagrangian density (2). We then follow the procedure outlined in [22] and allow the solitons to vary adiabatically through a slow variation of the soliton parameters with $Z = \epsilon z$ being the slow propagation variable. To first order in $\epsilon \ll 1$ the result is a set of dynamical equations governing the collective coordinates $x^{(i)}$, $y^{(i)}$, and $\phi^{(i)}$, being the center positions along the x and y-axis and accumulated phase of soliton i=1,2, respectively. We can express the new coordinates as $x^{(i)}(z) = \int_0^z \nu_x^{(i)}(Z')\mathrm{d}Z' + x_0^{(i)}$, $y^{(i)}(z) = \int_0^z \nu_y^{(i)}(Z')\mathrm{d}Z' + y_0^{(i)}$, and $\phi^{(i)}(z) = \int_0^z \lambda^{(i)}(Z')\mathrm{d}Z' + \phi_0^{(i)}$, where subscript 0 denotes initial values.

At this point one traditionally simplifies the system by assuming the velocities to be small ($\partial_z x^{(i)} \sim \epsilon$, $\partial_z y^{(i)} \sim \epsilon$), i.e. the solitons propagate almost in parallel. However, we are interested in velocities that can be considerable, so instead we assume symmetric interaction between in-phase solitons with initially identical profiles, $\lambda = \lambda^{(i)}$ and $P = P^{(i)}$, and equal but opposite velocities, $\nu_{x,y} = \nu_{x,y}^{(1)} = -\nu_{x,y}^{(2)}$. Without loss of generality we set $x_0 = x_0^{(1)} = -x_0^{(2)} \geq 0$ and $y_0 = y_0^{(1)} = -y_0^{(2)} \geq 0$. Symmetry is conserved and the two sets of collective coordinates degenerate to one, $X = x^{(1)} = -x^{(2)}, Y = y^{(1)} = -y^{(2)}$. In cylindrical coordinates with $R = \sqrt{X^2 + Y^2}$ we can then reduce the dynamical equations to the Euler-Lagrange equation of the effective Lagrangian

$$L(R, \dot{R}) = \frac{1}{2}P\dot{R}^2 - U_{\mathrm{eff}}(R, \dot{R}), \tag{5}$$

for the single coordinate R. The effective potential

$$U_{\mathrm{eff}}(R, \dot{R}) = \frac{C_0}{2R^2}P + \frac{1}{2}U(R, \dot{R}) \tag{6}$$

is composed of the classical centrifugal barrier, where $C_0 = (X\dot{Y} - Y\dot{X})^2 = (x_0\nu_y + y_0\nu_x)^2$ is constant because of conservation of angular momentum, and of the interaction integral

$$U = -\int V^{(1)}\left[V^{(1)}W^{(2)}\cos(2\phi) + 2W^{(1)}V^{(2)}\cos(\phi)\right]d\vec{r}_\perp \tag{7}$$

where $\phi = 2\nu_x x + 2\nu_y y$. We note that, strictly speaking, U is only a quasi-classical potential since it depends on the velocities (in contrast to the potential used in [22]).

We have now established a picture of an effective particle moving in a potential, U_{eff}, with the kinetic energy $E_{\mathrm{kin}} = \frac{1}{2}P\dot{R}^2 \geq 0$. For small velocities the potential (7) has the shape of a well and hence represents an attractive force. In the general case of non-planar interaction, $C_0 \neq 0$, the centrifugal barrier is always repulsive and

goes to infinity at $R=0$. This does not necessarily rule out fusion since also the velocities go to infinity because of conservation of C_0. The centrifugal barrier also creates a local minimum in the effective potential (still assuming small velocities) which suggests that spiraling configurations may exist. In general, however, we cannot expect our model to yield correct physical results in the vicinity of $R=0$ since it violates the assumption of weakly overlapping solitons. In fact fusion has been observed numerically, but stable spiraling configurations have not been found [22].

Here we shall not discuss the qualitatively different regimes. Rather we are interested in *quantitative* predictions of escape velocities. For solitons launched with outwards velocities we will all ways be able to theoretically predict the escape velocity. On the other hand, a consistent theory for the determination of the inwards escape velocity only exists for in-plane interaction, when the classical centrifugal barrier vanishes, i.e. $C_0 = 0$.

We first determine the outwards escape angle. For simplicity we focus on in-plane interaction with $y_0 = \nu_y = C_0 = 0$ wherefore $R = |X|$ and $\dot{R} = \nu_x$. In this case the effective particle either escapes the potential, $E_{\rm tot} = E_{\rm kin} + E_{\rm pot} > 0$, or is trapped by it, $E_{\rm tot} < 0$, and the escape velocity, ν_c, is given by the relation

$$\nu_c^2 P = U(x_0, \nu_c). \tag{8}$$

Unfortunately we are not able to express the interaction integral $U(x_0, \nu_x)$ in terms of analytical functions and we cannot use Gaussians, since the Gaussian tale asymptotic is different from that of the exact soliton. It is however trivial to solve (8) numerically and in Fig. 2 we have plotted the outwards escape angle, given by $\alpha_c = \mathrm{Arctan}(\nu_c)$, versus the phase mismatch. The initial beam width and separation are kept constant to ensure a weak overlap of the soliton tails at all phase mismatches.

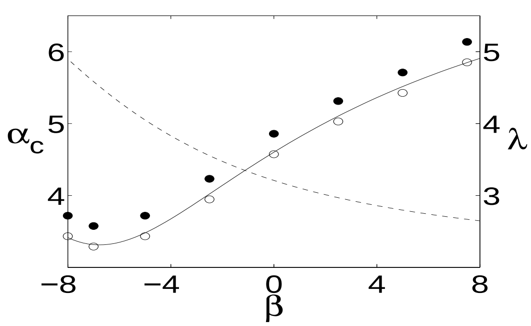

FIG. 2. Outwards escape angle in degrees (solid curve) for $FWHM$=1 and x_0=1.5. Numerical experiments where exact solitons fused ($\bullet$) and where they escaped ($\circ$). The dashed line shows the initial soliton power, P.

The simulations were performed with numerically found exact solitons as initial conditions. They confirm the accuracy of the escape angle predicted by Eq. (8). We found the minimum of about 3° around $\beta = -7$ to be global. Note that the angles are expected to be small, since the initial overlap and hence the attractive force between the solitons is weak. As an example of the dynamics we show in Fig. 1:C,D the outcome of the experiments with β=5 from Fig. 2. Only the fundamental waves are shown, the evolution of the SH waves being qualitatively the same.

Now considering solitons launched towards each other we first elaborate on the effective particle picture. If we assume the solitons to be initially far apart this corresponds to the effective particle experiencing essentially no potential. Even a small launch angle should then result in a positive total energy and enable the effective particle to cross the bottom of the potential and escape towards infinity, corresponding to soliton crossing. This is off course not the correct physical picture, since our system is not integrable and thus in reality the collision is not elastic. There is transfer of energy into internal soliton modes and shedding of energy as radiation.

In a different picture we assume that the soliton profiles do not change before the point of collision. This seems reasonable when comparing the characteristic length of slow adiabatic change with the relatively short interaction distance occurring for considerable velocities. In this case we can treat the interaction as if the solitons were launched on top of each other (x_0=0) corresponding to the effective particle being launched in the bottom of the potential, where it experiences the maximum barrier. Then the relation determining the escape velocity becomes $\nu_c^2 P = 2U(0, \nu_c)$ rather than (8). The interaction integral no longer depends on the asymptotic tales but on the entire profiles and hence we can apply the Gaussian approximation (3). The general transcendental equation for the inwards escape angle is then given by

$$\nu_c^2 = \frac{2}{b}\frac{\lambda b + 1}{2\lambda b - 1}\left[e^{-\frac{4}{3}b\nu_c^2} + 2e^{-\frac{1}{3}b\nu_c^2}\right], \tag{9}$$

which for β=0 simplifies to

$$\beta = 0: \quad \nu_c = 0.23 \times \sqrt{P}, \tag{10}$$

in terms of the power. In the large phase-mismatch cascading limit, ($\beta \gg \lambda$) where the nonlinearity is effectively cubic, Eq. (9) simplifies to

$$\beta \gg \lambda: \quad \nu_c = \sqrt{\frac{3}{4}\left(\frac{P}{2\pi} - \beta\right)}. \tag{11}$$

We remark that this approach is equivalent to finding the critical angle of total internal reflection for a waveguide [14]. However, since beam propagation in quadratic media does not induce changes in the refractive index the method used in [14] is not applicable to this case.

In Fig. 3 we have summarized the results for exact phase matching, β=0, and plotted the predicted inwards escape angle, $\alpha_c = \mathrm{Arctan}(\nu_c)$, versus soliton power, both for ν_c given by (10) and for ν_c found with exact solitons. The curves are close and the simple square

root dependency on the power excellently predict the escape angle. In the experiments we used Gaussians as initial conditions. These were launched with a distance of $2x_0$=10 between them, ensuring practically zero initial overlap. In Fig. 1:A,B we show examples of experiments with $\beta = 0$ and a power of $P = 48.3$. We note that for the exact soliton initial conditions we observed even better agreement than with Gaussians. Close to the escape angle all the power is shed as radiation and thus (10) serves as an *accurate prediction of soliton annihilation* (Fig. 4:B).

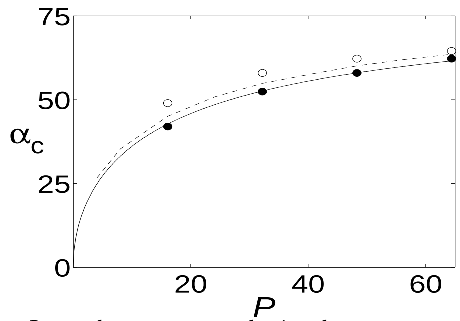

FIG. 3. Inwards escape angle in degrees versus soliton power calculated analytically with Gaussians (solid) and with exact solitons (dashed) for β=0 and x_0=5. Numerical experiments where Gaussians crossed (○) and fused (●).

We also investigated the cases of non-zero mismatches, focusing on $\beta = \pm 3$. In these regimes there is a power threshold for soliton excitation and the collisions are of a much more complex nature than for perfect phase matching, where solitons exist at all powers. For relatively low powers not far above the threshold the collisions mostly resulted in destruction of the solitons (Fig. 4:A). The explanation of this phenomenon is that too much power is shed as radiation in the collision and hence the resulting beams diffract because they do not carry sufficient power to form solitons. For higher powers the predicted escape angles were reasonably close to the observations.

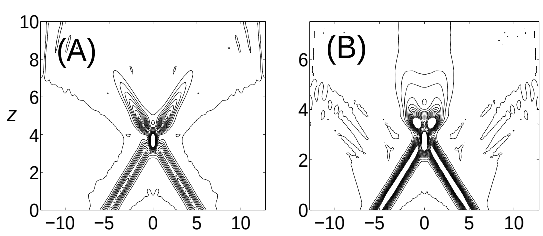

FIG. 4. A: Crossing followed by diffraction ($\beta = -3$ and $P = 45.3$). B: Annihilation ($\beta = 0$ and $P = 48.3$).

In conclusion we have developed a theoretical description that should hold for systems with all types of local nonlinearities. In particular we have studied bulk $\chi^{(2)}$ media and determined analytical expressions for the escape angles when the centrifugal barrier vanishes. This happens in the two in-plane cases of outwards and inwards launched solitons. The simple expression for the inwards escape angle represents the first *analytical* prediction of the *outcome* of a soliton collision. We have verified the analytical expressions numerically using Gaussian approximations and observed excellent agreement.

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